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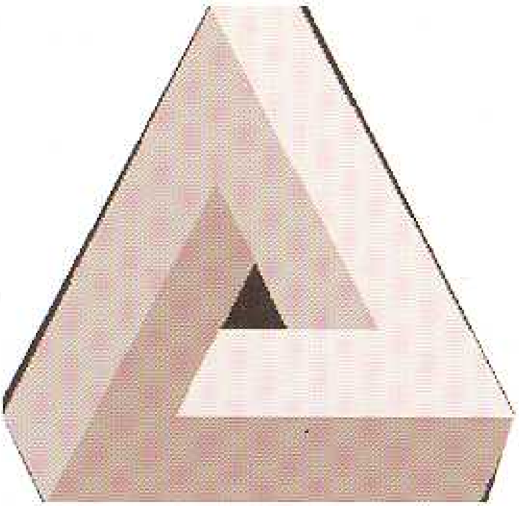

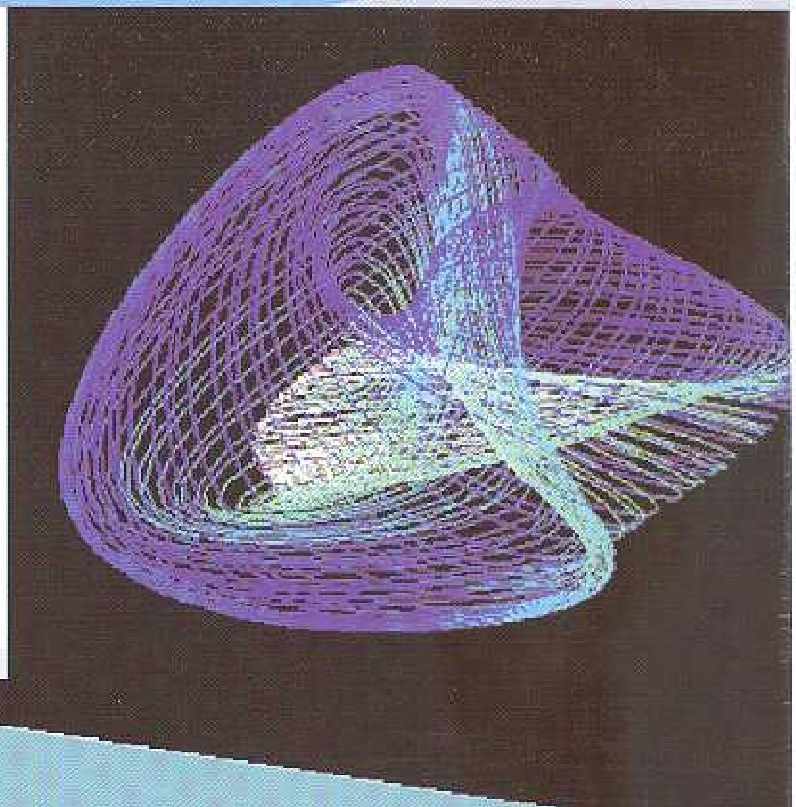

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CONTENTS

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Exploring Bi-Criteria versus Multi-Dimensional Lower Partial Moment Portfolio Models

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ABSTRACT: This contribution explores how multi-dimensional lower partial moment portfolio models are different from their bi-criteria counterparts. In particular, the mean semi-variance and semi-skewness model that seems little used in practice is contrasted to the rather popular mean semi-variance and mean semi-skewness models. The difference between these models is illustrated via a geometric reconstruction of the multi-dimensional efficient portfolio choice set.

Keywords: Lower Partial Moments, Efficient Frontier, Mean-Variance-Skewness Efficiency, Semi-Variance, Semi-Skewness.

1. INTRODUCTION

Markowitz (1952) and Roy (1952) provided investors with new quantitative decision models allowing allocating assets by considering the optimal trade-off between return and some measure of risk. Following these seminal works, modern portfolio theory models the fundamental portfolio optimization problem as a bi-criteria mean-risk problem where expected return is maximized and some (scalar) risk measure is minimized. In the original Markowitz (1952) mean-variance (MV) model the risk is measured by the variance and this has become the dominant model, despite its strong assumptions regarding investor preferences and underlying asset distributions. But, in the meanwhile, several other risk measures have been considered yielding an entire family of bi-criteria mean-risk models (see, e.g., the survey of Constantinides and Malliaris (1995) or Wang and Xia (2002)).

Roy's (1952) safety first technique assumes that investors prefer safety of the principal first and therefore fix a minimal acceptable return (disaster level) conserving the principal. Investors would then prefer investments with the smallest probability of going below the disaster level. This safety first technique has been instrumental in the development of downside risk measures. Indeed, already Markowitz (1959: Chapter 9) recognized that investors are interested in minimizing downside risk for two main reasons: (1) only downside risk or safety first is relevant to an investor, and (2) security distributions may not be normally distributed. Hence, downside risk measures are more helpful in making proper decisions in case of non-normal security return distributions. He offers two ways of measuring downside risk: (i) a semi-variance computed from the mean return (below-mean semi-variance), and (ii) a semi-variance computed from a target return (below-target semi-variance).

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The idea of using downside risk measures rather than the variance has led to a whole series of developments (see the historical survey paper by Nawrocki (1999)). For instance, Hogan and Warren (1974) developed a below-target semi-variance capital asset pricing model (CAPM). While the basic CAPM model assumes asset distributions are normal, this variation is of interest when distributions are non-normal and asymmetric. More general lower partial moment CAPM models have later on been presented in Harlow and Rao (1989), among others. Meanwhile, other downside risk measures have also been proposed in the literature: for instance, value at risk (VaR), conditional VaR, etc.

Bawa (1975) is the first author defining lower partial moment (LPM) as a general family of below-target risk measures. He also has proven that LPM measures are related to stochastic dominance. Bawa (1978) and Fishburn (1977) generalize the LPM models by accounting for both different levels of investor risk tolerance and by allowing for a target return. They demonstrated that, irrespective of any distributional restrictions (in contrast to the MV-criterion), dominance in terms of mean-LPM models is necessary for stochastic dominance of a given order and hence expected utility (EU) maximization. Thus, mean lower partial moments (LPM) models always satisfy the necessary (though not the sufficient) conditions for EU theory.

While downside risk measures are around for quite a moment, theoretical developments may have been partially hampered by lack of computing power (see Nawrocki (1999)). Furthermore, it took quite some time before the invalidity of the assumptions underlying the traditional mean variance model became widely acknowledged. Finally, research in behavioral finance has abundantly documented investor characteristics like downside loss aversion (e.g., Kahneman and Tversky (1979)) which cannot be accounted for in the traditional portfolio models. These developments can explain the recent upsurge in the academic literature on LPM measures (see, for instance, Estrada (2002), Galagedera (2007), Jarrow and Zhao (2006), Jin *et al.* (2006)).

According to Nawrocki (1999), around 1990, downside risk measures have made their appearance in the practitioner literature. One rather well-known example is the introduction of the Sortino ratio based on a comparison with a target return and semi-variance (see Sortino and van der Meer (1991)). Recently, the interest in portfolio management and evaluation for LPM measures has been rekindled for various reasons. First, in the financial industry capital determination is a hot topic due to numerous crises and the implementation of the Basel regulations. Second, foreign currency and equity derivatives have become popular in managing equity portfolios, but these have the potential to induce non-symmetries in the equity portfolio's distribution. Other reasons are listed in, e.g., Jarrow and Zhao (2006).

However, it is sometimes ignored in this literature that mean LPM models do impose strong assumptions on investor preferences. Fundamentally, investor utility should only depend on the mean and the partial moment appearing in the specific bi-criteria problem selected (for instance, mean and semi-variance, or mean and semi-skewness, etc.). Otherwise, just like in the general moment case (see Samuelson (1970)), additional LPM may have to be included to better approximate the underlying investor utility functions. Konno et al. (1993), for instance, have been adding a lower semi-skewness to a given mean lower semi-variance model yielding a three dimensional LPM model.

The purpose of this contribution is to explore the geometric structure of the mean lower semi-variance and lower semi-skewness models to shed some light on common misunderstandings sometimes found in the LPM literature. The widely used mean-semi-variance and mean-semi-skewness models are just particular cases of a larger set of efficient portfolios generated by the MVS lower partial moment frontier à la Konno *et al.* (1993). In particular, mean-semi-variance and mean-semi-skewness models are lower respectively upper boundaries in the skewness dimension of the latter general model.

For that purpose, we employ the shortage function framework (see Luenberger (1995)) introduced in portfolio theory by Briec *et al.* (2004) as a way to characterize the MV portfolio choice set. These authors integrate the shortage function into the basic MV Markowitz model and develop a dual framework to assess the degree of satisfaction of investors' preferences. Furthermore, they decompose portfolio performance into portfolio and allocative efficiency. Moreover, via the shadow prices associated with the efficiency measure, duality yields information about investors' risk aversion. This work has been extended in Briec *et al.* (2007) to the non-convex mean-variance-skewness (MVS) space.

Thus, in line with other micro-economic fields like consumption and production theory, the boundary of a possibly multi-dimensional and non-convex multi-moment portfolio choice set can be characterized by a distance function (Cornes (1992)). This same function has an efficiency interpretation allowing it to serve as a useful benchmark. This shortage function is furthermore compatible with rather general investor preferences via its duality to possibly multi-moment utility functions.

The structure of the contribution is as follows. Section 2 develops the theoretical framework based upon the shortage function as described in Briec *et al.* (2004, 2007). Section 3 defines the LPM. The next section describes the data set and some of its characteristics. Empirical results are reported in Section 5. A final section concludes.

2. THE GENERAL MOMENT MVS PORTFOLIO FRAMEWORK

In this section, we briefly sketch the general Mean-Variance-Skewness (MVS) portfolio framework that is adapted in the next section to lower partial moments. Basically, this framework follows the theory developed by Briec *et al.* (2007), which is an extension of the non-parametric approach to Mean-Variance (MV) portfolio optimization initiated by Briec *et al.* (2004).

Assume n financial assets (or funds) are given by their historical returns over a specified time window of length m. A portfolio (or fund of funds) $x = (x_1, \ldots, x_n)$ selected from these assets is a vector for which x_i, $(i = 1, \ldots, n)$ represents the proportion of the corresponding asset, with $\sum_{i=1}^{n} x_i = 1$. If we exclude short sales, then all x_i are non-negative. But, this hypothesis can be easily relaxed. In general, the portfolio simplex is the subset of $\mathbb{R}^n$ determined by

$$\Im = \left\{ x \in \mathbb{R}^n \,\middle|\, \sum_{i=1}^{n} x_i = 1, x_i \geq 0 \right\}.$$

The expected return vector, covariance matrix and coskewness tensor of the n financial assets can be computed. More precisely, if r_{il} denotes the historical return of the i^{th} financial asset ($i = 1, \ldots, n$) on time l ($l = 1, \ldots, m$), then the expected return of the i^{th} asset over the given time window is equal to

$$R_i = \frac{1}{m} \sum_{l=1}^{m} r_{il}. \tag{1}$$

Furthermore, the covariance between the i^{th} and the j^{th} asset, denoted by V_{ij}, is obtained as

$$V_{ij} = \frac{1}{m} \sum_{l=1}^{m} (r_{il} - R_i)(r_{jl} - R_j), \tag{2}$$

and the coskewness between the i^{th}, j^{th} and k^{th} asset, denoted by S_{ijk}, is computed as

$$S_{ijk} = \frac{1}{m} \sum_{l=1}^{m} (r_{il} - R_i)(r_{jl} - R_j)(r_{kl} - R_k). \tag{3}$$

Now, the expected return $\mathrm{E}[R(x)]$ of this portfolio x, its variance $\mathrm{Var}[R(x)]$ and skewness $\mathrm{Sk}[R(x)]$ are calculated as follows:

$$\mathrm{E}[R(x)] = \sum_{i=1}^{n} x_i R_i, \quad \mathrm{Var}[R(x)] = \sum_{i,j=1}^{n} x_i x_j V_{ij}$$

$$\mathrm{Sk}[R(x)] = \sum_{i,jk=1}^{n} x_i x_j S_{ijk}. \tag{4}$$

Notice that this formulation is expressed in terms of expected returns. However, it is equally possible to employ historical return information and to gauge performance ex-post or retrospectively rather than ex-ante. In the empirical part of this contribution, we actually use such historical returns for illustrative purposes. Notice that in view of the efficient market hypothesis (in particular, when asset price motions are modeled using random walk models, or when forecasts of the risk premium and the risk free rate are not available for use in an equilibrium model), one can view historical returns as a simplified mechanism to generate expected return information. In a similar vein, we maintain the hypothesis of historical volatility stability instead of using stochastic volatility models or implied volatility derived from option pricing models. This same logic also applies to the higher moment information employed in this research. We consider that the probable bias introduced by these simplifications (see, for example, Elton (1999)) is acceptable in this contribution where the validity of the empirical illustration does not rely upon these approximations. Obviously, drastic improvements in determining these expected moments should be achieved prior to any portfolio management task.

The portfolio simplex $\Im$ is mapped into the 3-dimensional MVS-space by means of the function $\Phi : \Im \to \mathbb{R}^3 : x \mapsto \Phi(x) = (\mathrm{E}[R(x)], \mathrm{Var}[R(x)], \mathrm{Sk}[R(x)])$. The image set $\Phi(\Im)$ is now extended to what is called the MVS disposal representation set $\mathrm{DR} = \Phi(\Im) + (\mathbb{R}_- \times \mathbb{R}_+ \times \mathbb{R}_-)$.

Briec *et al.* (2007) then define the weakly efficient MVS-frontier, a subset of the portfolio representation set, as

$$\partial^W(DR) = \{(E, V, S) \in DR \mid (-E', V', -S') < (-E, V, -S) \Rightarrow (E', V', S') \notin DR\}.$$

As the notation suggests, the weakly efficient MVS-frontier is part of the boundary of the disposal representation set. As was observed by Kerstens *et al.* (2008), the disposal representation set itself is an extension of $\Phi(\Im)$. Consequently, the weakly efficient MVS-frontier can contain points that are not attainable by an admissible portfolio.

The strongly efficient MVS-frontier is introduced as

$$\partial^S(DR) = \{E, V, S) \in DR \mid (-E', V', -S') \leq (-E, V, -S) \text{ and } (-E', V', -S') \neq (-E, V, -S) \Rightarrow (E', V', S') \notin DR\}.$$

This subset is in its turn a part of the weakly efficient MVS-frontier and contains only points that can actually be reached by true portfolios.

Given a direction vector $g = (g_R, g_V, g_S) \in \mathbb{R}_+ \times \mathbb{R}_- \times \mathbb{R}_+$, the shortage function S_g with respect to g is defined by $Sg(y) = \sup\{\delta \mid y + \delta g \in DR\}$, for any point y in MVS–space. In particular, y can belong to DR. The choice of direction vector is discussed below.

Practically, the shortage function is computed for a given MVS point $y = (y_R, y_V, y_S)$ by solving the following cubic non-linear programming model:

$$\begin{aligned} \max_{x,\delta}\ \delta \text{ s.t. } & \sum_{i=1}^{n} x = 1, \\ & \mathrm{E}[R(x)] \geq y_R + \delta g_R, \\ & \mathrm{Var}[R(x)] \leq y_V + \delta g_V, \\ & \mathrm{Sk}[R(x)] \geq y_S + \delta g_S. \end{aligned} \tag{P1}$$

Notice that dropping the last skewness constraint allows computing the shortage function for a mean-variance model.

Furthermore, it was shown by Briec *et al.* (2007) that the shortage function can be interpreted in a dual, mean-variance-skewness based framework. For that purpose, one needs the notions of a direct and an indirect utility function. For given investor's preferences μ, ρ and κ ($\frac{\rho}{\mu}$ represents the absolute risk aversion and $\frac{\kappa}{\mu}$ the absolute prudence), and some portfolio $x \in \Im$, the direct MVS utility function is given by

$U(x;\mu,\rho,\kappa) = \mu E[R(x)] - \rho Var[R(x)]\ \kappa Sk[R(x)]$.

When maximized over all portfolios, it yields the indirect MVS utility function $U^*(\mu,\rho,\kappa)$.

With these utility functions, it is possible to define an efficiency taxonomy consisting of the notions of overall efficiency (OE), allocative efficiency (AE) and portfolio efficiency (PE).. Turning to the definition of each of these three components, one finds:

$$OE(x;\mu,\rho,\kappa) = \frac{U^*(\mu,\rho,\kappa) - U(x;\mu,\rho,\kappa)}{\mu g_R - \rho g_V + \kappa g_S},$$

$$AE(x;\mu,\rho,\kappa) = OE(x;\mu,\rho,\kappa) - S_g(x) \text{ and}$$

$$PE(x) = S_g(x).$$

It follows that the overall efficiency can be decomposed in an additive way as follows:

$$OE(x;\mu,\rho,\kappa) = AE(x;\mu,\rho,\kappa) - PE(x).$$

We add two additional remarks. First, notice that differences between shortage functions of models containing different dimensions can serve as a test to verify the impact of adding one or more moments (see Briec *et al.* (2007): Remark 3.5). This is simply a consequence of the fact that the objective function in (P1) remains the same irrespective of whether a skewness constraint is added or not. Hence, the maximal value function is normally non-increasing when adding constraints. Second, remark that the allocative efficiency used here also includes the so-called convexity efficiency introduced in Briec *et al.* (2007).

3. LOWER PARTIAL MOMENTS IN THE MVS FRAMEWORK

The framework introduced in section 2 is based upon a record of historical returns and their computed first, second and third moments (as given by (1), (2) and (3)). Instead of using these general moments, several alternatives can be proposed. In this contribution, we focus on the lower partial moments.

More precisely, we replace the covariance given in (2) by the lower semi-covariance

$$V_{ij}^- = \frac{1}{m}\sum_{l=1}^{m} \min(r_{il} - R_i, 0)\min(r_{jl} - R_j, 0), \quad (5)$$

and the coskewness given in (3) by the lower semi-coskewness

$$S_{ijk}^- = \frac{1}{m}\sum_{l=1}^{m} \min(r_{il} - R_i, 0)\min(r_{jl} - R_j, 0)\min(r_{kl} - R_k, 0). \quad (6)$$

It follows immediately that the lower semi-covariances are all positive, whereas all lower semi-coskewnesses are negative.

Notice that in the literature (e.g., Nawrocki (1999)), one can find alternative definitions of the lower partial moments that basically yield asymmetric co-semi-variances and so on (rather than symmetric ones as in the above definitions and following, e.g., Estrada (2002)). Obviously, these alternative symmetric or asymmetric definitions yield different results (as documented by, for instance, Sing and Ong (2000)).

For the return, variance and skewness of the portfolio x, we use the same equations as given in (4) above, but now written in terms of lower partial moments. Thus, we obtain:

$$E[R(x)] = \sum_{i=1}^{n} x_i R_i, \quad Var[R(x)] = \sum_{i,j=1}^{n} x_i x_j V_{ij}^-,$$

$$Sk[R(x)] = \sum_{i,j,k=1}^{n} x_i x_j S_{ijk}^-. \quad (7)$$

We remark that the semi-variance and semi-skewness introduced in (7) do not possess the same statistical properties as their general counterpart given in (4). Indeed, the latter equations can be derived mathematically from the fact that the return of a portfolio is expressed as a linear combination of the returns of the individual assets, with the portfolio weights as coefficients. Although this is still valid when working with lower semi-variance and lower semi-skewness (this has no influence on the returns), the mathematical deduction to obtain the variance and skewness of the portfolio is no longer valid.

Therefore, the equations in (7) are considered as definitions in the case of working with lower partial moments.[4]

4. SAMPLE FROM THE FRENCH EURONEXT-NYSE CAC40 INDEX

We now turn to an investigation of the impact of working with lower partial moments when implemented in the MVS framework. After different experiments with several datasets (including artificial data), we have chosen to present empirical findings in this contribution by means of the following dataset. The data consists in 27 blue-chips stocks included in the French Euronext-NYSE CAC40 index.[5] These data were collected on a monthly basis using the so-called closing price for each of these stocks. The time window over which these data were collected ranges from the first quotation day in January 1988 to the first quotation day in March 1996 (99 months, or 8 years and 3 months). We then converted these prices in continuous returns using the standard transformation

$$R_{i,t} = \ln(p_{i,t}) - \ln(p_{i,t-1}),$$

with the monthly return of stock i over month t–1 and $p_{i,t}$ the closing price for the first day of quotation in month t for stock i. We thus obtain 98 observations per time-series.

Table 1
General Moments and Shapiro-Wilk Test for Normality

Nr	*Stock*	*Ret.*	*Var.*	*Skew.*	*Shapiro-Wilk Statistic*	*p-value*	*Sig*
1	BRL	0.0838	0.7274	0.1648	0.9870	0.4506	
2	AIR	0.1027	0.3334	0.0285	0.9952	0.9818	
3	CGE	0.0773	0.5935	0.0043	0.9849	0.3277	
4	MIDI	0.0585	0.8904	0.6243	0.9509	0.0011	***
5	BYG	0.0120	0.7989	0.3171	0.9716	0.0323	**
6	CAN	0.0950	0.5194	-0.2843	0.9596	0.0043	***
7	CGS	-0.0010	0.9000	0.2884	0.9868	0.4380	
8	CRFR	0.2060	0.4531	-0.0190	0.9949	0.9734	
9	CSO	0.0431	0.7997	-0.1314	0.9878	0.5057	
10	CCF	0.0832	0.5431	-0.1101	0.9715	0.0314	**
11	BSN	0.0725	0.2887	-0.0115	0.9948	0.9708	
12	ORAF	0.1771	0.4724	0.1226	0.9877	0.5026	
13	LFG	0.0388	0.6573	-0.2074	0.9841	0.2874	
14	LVMH	0.1519	0.5270	-0.0409	0.9452	0.0005	***
15	MCL	0.0407	1.0336	-0.1582	0.9879	0.5154	
16	PGT	0.0556	0.6676	-0.0318	0.9929	0.8888	
17	PRNT	0.1214	0.8311	0.0796	0.9892	0.6161	
18	GOB	0.0487	0.5312	-0.1419	0.9833	0.2497	
19	SQAF	0.1159	0.5402	-0.0050	0.9845	0.3036	
20	QTAF	0.1320	1.1093	0.0371	0.9862	0.3990	
21	SGE	0.0693	0.5728	0.0561	0.9923	0.8510	
22	SDX	0.1729	0.6073	-0.0621	0.9715	0.0314	**
23	LE	0.0651	0.7052	-0.0953	0.9812	0.1755	
24	CSF	-0.0087	0.8090	0.1477	0.9909	0.7505	
25	CFP	0.1374	0.4624	0.0670	0.9935	0.9205	
26	VAL	0.1219	0.7774	-0.6872	0.9470	0.0006	***
27	EXAF	0.0806	0.5344	-0.0987	0.9866	0.4282	

H_0: Distribution of returns for stock X is normally distributed.
* Rejection at 10% level, **, 5%, *** 1%

We test these data for normality of the underlying distribution by performing the Shapiro-Wilk test for normality (see Table 1). Although this test is not very powerful, it clearly reveals that a large majority (75%) of these assets actually fit the Gaussian distribution (except 7 among 27). This is more or less in line with theoretical predictions that severe deviations from normality are more frequent at the intraday or daily level, but that monthly data tend to converge towards the normal distribution. One can also notice that 15 asset return distributions exhibit negative skewness. In other terms, the downside changes for these assets vary over a broader magnitude than the upswings. One can imagine that these assets could play a role in multidimensional LPM portfolio selection.

5. EMPIRICAL RESULTS

5.1. Shortage Function Results

We start by computing the value of the shortage function for all 27 stocks both in the MV- and the MVS-framework, and for lower partial and general moments. The results for the lower partial and general moments are reported in the second and third part of Table 2, respectively. In each of these parts, one finds first a column with the MV model (δ_{MV}) and thereafter one containing the MVS model (δ_{MVS}). The final column computes the difference between both shortage functions ($\Delta\delta$) to obtain an idea of the impact of adding the third moment. The direction vector g necessary to compute this value is chosen as indicated in

Table 2
Comparison of Shortage Function Values

		LPM-statistics			*LPM*			*General Moments*		
Nr	*Stock*	*Ret.*	*Var.*	*Skew.*	δ_{MV}	δ_{MVS}	$\Delta\delta$	δ_{MV}	δ_{MVS}	$\Delta\delta$
1	BRL	0.0838	0.3414	-0.4632	0.5945	0.5945	0.0000	0.7037	0.3716	0.3321
2	AIR	0.1027	0.1603	-0.1432	0.1750	0.1750	0.0000	0.3625	0.2267	0.1357
3	CGE	0.0773	0.3033	-0.4020	0.5637	0.5637	0.0000	0.6588	0.6146	0.0442
4	MIDI	0.0585	0.3836	-0.5228	0.6743	0.6743	0.0000	0.7836	0.0000	0.7836
5	BYG	0.0120	0.3789	-0.4907	0.6709	0.6709	0.0000	0.7591	0.0000	0.7591
6	CAN	0.0950	0.3127	-0.4901	0.5354	0.5354	0.0000	0.5706	0.5706	0.0000
7	CGS	-0.0010	0.4176	-0.5942	0.7013	0.7013	0.0000	0.7862	0.3530	0.4331
8	CRFR	0.2060	0.2279	-0.2538	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
9	CSO	0.0431	0.4258	-0.6588	0.7071	0.7071	0.0000	0.7593	0.7593	0.0000
10	CCF	0.0832	0.2735	-0.3935	0.5085	0.5085	0.0000	0.6178	0.6178	0.0000
11	BSN	0.0725	0.1460	-0.1269	0.1459	0.1459	0.0000	0.3334	0.3144	0.0190
12	ORAF	0.1771	0.2137	-0.2050	0.0967	0.0907	0.0060	0.1356	0.0000	0.1356
13	LFG	0.0388	0.3610	-0.5471	0.6545	0.6545	0.0000	0.7072	0.7072	0.0000
14	LVMH	0.1519	0.2674	-0.4479	0.2781	0.2781	0.0000	0.2919	0.2897	0.0022
15	MCL	0.0407	0.5368	-0.9483	0.7677	0.7677	0.0000	0.8138	0.8138	0.0000
16	PGT	0.0556	0.3324	-0.4398	0.6248	0.6248	0.0000	0.7117	0.7048	0.0069
17	PRNT	0.1214	0.4006	-0.5980	0.5465	0.5465	0.0000	0.5822	0.4571	0.1251
18	GOB	0.0487	0.2871	-0.3907	0.5655	0.5655	0.0000	0.6377	0.6377	0.0000
19	SQAF	0.1159	0.2733	-0.3420	0.4220	0.4220	0.0000	0.5004	0.4706	0.0298
20	QTAF	0.1320	0.5358	-0.8721	0.5614	0.5614	0.0000	0.5614	0.0000	0.5614
21	SGE	0.0693	0.2787	-0.3299	0.5435	0.5435	0.0000	0.6589	0.4973	0.1616
22	SDX	0.1729	0.3038	-0.4665	0.1914	0.1914	0.0000	0.1914	0.1914	0.0000
23	LE	0.0651	0.3800	-0.5582	0.6641	0.6641	0.0000	0.7243	0.7243	0.0000
24	CSF	-0.0087	0.3877	-0.5060	0.6783	0.6783	0.0000	0.7621	0.4926	0.2695
25	CFP	0.1374	0.2175	-0.2133	0.2495	0.2495	0.0000	0.3383	0.0000	0.3383
26	VAL	0.1219	0.4763	-1.0290	0.5944	0.5944	0.0000	0.5644	0.5644	0.0000
27	EXAF	0.0806	0.2884	-0.3439	0.5365	0.5365	0.0000	0.6181	0.6181	0.0000

Briec *et al.* (2004, 2007). This implies that it is determined by the position of the financial product under observation. In the case of the MVS-framework, for instance, $g = (| E[R(x)] |, -Var[R(x)], | Sk[R(x)] |)$ for a particular portfolio x. These direction vectors can be derived from the first part of Table 1 (Table 2) for the general (partial) moments.

Studying Table 2, two main results appear. First, for the general moments, the impact of adding the skewness is clearly not zero on average. The difference between the densities of the shortage function values obtained with the MV and MVS models can be tested with a statistic developed by Li (1996) and later refined by Fan and Ullah (1999). This test is valid for both dependent and independent variables that asymptotically follow a standard normal distribution. The null hypothesis states the equality of both distributions. Computing this test statistic results in the value 1.44. Consequently, the equality of these efficiency distributions can be safely rejected at a significance level of 10% (1.28 is the critical value). Thus, there is a statistical indication that both efficiency measures follow a different distribution, or put differently, adding skewness in the case of general moments indeed makes a difference.

Second, in the case of lower partial moments, except for one stock (stock number 12 ORAF), all computed values of the shortage function for both MV and MVS models are identical. For many other datasets we have examined, including randomly generated data, no differences were detected at all. This surprising and unexpected result indicates that in the large majority of cases, the semi-skewness constraint in model (P1) is non-binding. If it was not for the minor, but significant difference observed in stock number 12 ORAF (a difference which is not related to numerical issues), one could even suggest that the MVS-framework is identical to the MV-framework. Computing the Li test statistic results in the value 2.15E-15, implying that the hypothesis that MV and MVS LPM models are identical cannot be rejected.

5.2. Geometric Reconstruction of the Multi-dimensional LMP Model

To investigate this puzzling phenomenon more profoundly, we decide to generate a geometrical representation of the weakly and strongly efficient frontiers using the guidelines set out in Kerstens *et al.* (2008). We recall that these authors reconstructed both the weakly and strongly efficient frontier geometrically by exploring several methods (shortage function and a variation of the same function, different grid strategies, etc.). Extensive comparison of different techniques shows that the shortage function leads to superior results, especially when applied on two-dimensional grids of fictitious assets positioned parallel to the coordinate planes, and projected in a direction perpendicular to the corresponding plane.

We clarify this conclusion that defines the setup for this study. Position, for instance, a plane parallel to the MV-plane at a very low skewness level. In this plane, a square grid can be considered for which the grid points can be seen as fictitious assets. Indeed, it might well be that these points are not contained in the image set $\Phi(\mathfrak{I})$. Nevertheless, such a point can be projected by means of the shortage function in a direction perpendicular to the plane containing the grid, this is, in the skewness direction. Therefore, model (P1) needs to be solved. If this does not lead to infeasibility, then a point from the weakly and strongly efficient frontier can be extracted from the solution. This process is now repeated for all points in the grid. Additionally, this procedure can be repeated for several grids parallel with the mean-skewness (MS)- and the variance-skewness (VS)-planes, leading to additional points of the geometrical reconstruction.

Figure 1 shows the results of such a geometric reconstruction yielding the weakly and strongly efficient frontiers computed with lower partial moments. In Figure 1, the weakly efficient MVS lower partial moments frontier is labeled ①, whereas ② points to the strongly efficient MVS-frontier. Notice that the weakly efficient frontier has horizontal and vertical extensions enlarging the strongly efficient frontier. The weakly efficient frontier contains portfolios that are of little interest to investors (since these are dominated in one or more dimensions). This concept is of value in reconstructing the strongly efficient frontier (see Kerstens *et al.* (2008)). At first glance, the strongly efficient MVS-frontier looks like a curve in MVS-space.

However, this turns out to be wrong. When zooming in substantially (as done on the right part of Figure 1 indicated by the black arrow), the strongly efficient MVS-frontier appears to be a narrow surface (in the form of a strip) rather than a simple curve. In fact, the dimension of the two-dimensional grid needed to be increased sufficiently to detect this strip. In particular, a grid size of 200 × 200 fictitious assets was taken.

Similar to the observations in Kerstens *et al.* (2008) in the case of general moments, the strongly efficient MV-frontier labeled ③ in the zoomed part of Figure 1 determines a lower boundary of the strongly efficient MVS-frontier. This clearly indicates that the widely used mean-semi-variance model is just a particular case of a larger set of efficient portfolios of the MVS lower partial moments frontier.

Furthermore, the strongly efficient MS-frontier labeled in the zoomed part of Figure 1 determines an upper boundary of the strongly

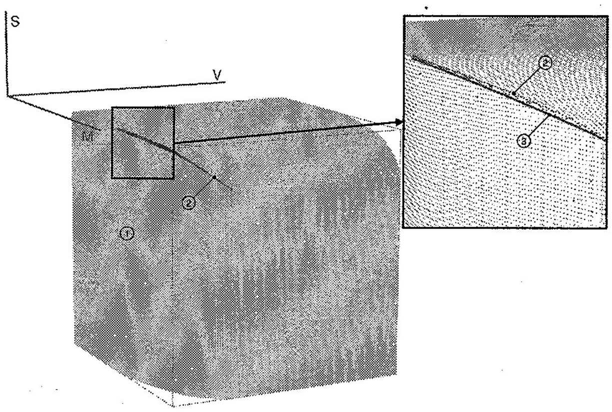

Figure 1: Weakly and Strongly Efficient Frontiers Computed with Lower Partial Moments.

efficient MVS-frontier. Again, this reveals that the mean-semi-skewness model is just a particular case of a larger set of efficient portfolios of the MVS lower partial moments frontier.

Together, both observations underscore the claim made in the introduction that the traditional bi-criteria LPM models are special cases of a more general, multidimensional partial moments frontier. Consequently, the model of Konno *et al.* (1993) and its further obvious generalizations probably merit more attention in the future.

Now the question arises why the strongly efficient MVS-frontier, when computed for lower partial moments, seems so small and narrow. We know that the strongly efficient MVS-frontier is a part of the weakly efficient frontier, which in its turn is the boundary of the disposal representation set *DR*. This set is an extension of the image set $\Phi(\Im)$. Consequently, it is this set that determines the shape of the strongly efficient MVS-frontier. Thus, to answer this question, we need to find out how the boundary of $\Phi(\Im)$ looks like and what determines the shape of this boundary.

Let us first point out that visualizing the boundary of $\Phi(\Im)$ itself is very difficult, if not impossible, because of the complexity of this set. This is the main reason why one introduces extensions of this set. However, we can visualize at least the interesting part of the boundary $\Phi(\Im)$ quite reasonably in the following way. Inspired by Kerstens *et al.* (2008), we start again with 2-dimensional grids of fictitious assets positioned parallel with the coordinate axes and project these orthogonally. The projection itself is performed by solving a cubic non-linear programming model. We mention in (P2) below the model that is solved for a 2-dimensional grid parallel with the MV-plane:

$$\max_{x,\delta} \delta \ s.t. \quad \sum_{i=1}^{n} x = 1, \quad E[R(x)] = y_R, \quad Var[R(x)] = y_V, \quad Sk[R(x)] = y_s + \delta |y_S|. \tag{P2}$$

In this model, $y = (y_R, y_V, y_S)$ represents one of the grid points. Similar models can be formulated for other coordinate planes.

In fact, model (P2) is derived from model (P1) by changing the inequalities to equalities and choosing the appropriate direction vector. As can be seen easily, model (P2) is equivalent with model (P3) given below, from which it follows immediately that the solution, if it exists, is part of the boundary of $\Phi(\mathfrak{I})$.

$$\max_x \text{Sk}[R(x)] \text{s.t.} \quad \sum_{i=1}^{n} x = 1, \quad \text{E}[R(x)] = y_R, \quad \text{Var}[R(x)] = y_V, \tag{P3}$$

For clarity, from a mathematical point of view, these models (P1) to (P3) can lead to infeasibilities for certain grid points, and, due to its non-convex nature, several local optima may well appear for other grid points. In this respect, the solution obtained needs not be situated at the 'outer boundary', which is the one determined by the global optimum. This disadvantage, however, does not pose large problems when it comes to visualization. Indeed, generating sufficient MVS-points can give us a fairly good idea of the shape.

For reasons of comparison, we create an image of the boundary of $\Phi(\mathfrak{I})$ as described above, for both the general moments and the lower partial moments. In both cases, 2-dimensional grids of size 50 × 50 are projected. Figure 2a shows the visualization when lower partial moments are used and Figure 2b presents the boundary of $\Phi(\mathfrak{I})$ in the case of general moments. Notice that, for the sake of comparison, the three axes are presented from exactly the same angle. However, since the semi-skewness is always negative, it follows that the LPM frontier is inevitably situated below the MV plane (which need not be the case for the general moments). In both cases, a triangulation between the boundary points is computed, which improves the visual interpretation of the set. Figure 2c and Figure 2d provide two dimensional views of the corresponding portfolio sets. More specifically, these are projections into the VS-plane.

Also in Figure 2, both the triangulations and the points themselves have a grayscale value according to the number of assets the final portfolios actually contain. A lighter color means that more assets are active in the optimal solution. In this way, strategies of diversification versus concentration can be observed. Focusing on the lighter gray tones, it seems clear that from the current viewing angle the general moments yield more diversified solutions compared to the LPM.

Especially Figure 2c and Figure 2d show a remarkable difference in the outer shape of the two sets. In the case of lower partial moments, the shape is more narrow and pointed left upwards in the VS-plane, while there is no obvious direction of preference when general moments are used. Thus clearly, there is a much stronger relationship between the lower semi-variance and the lower semi-skewness than is the case between variance and skewness.

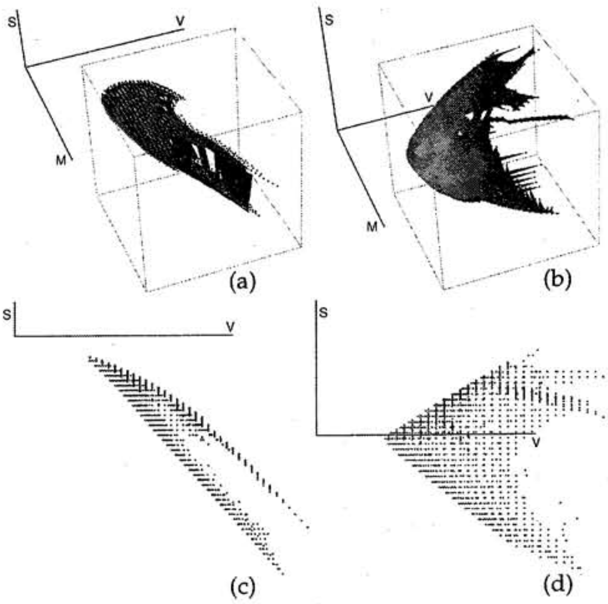

Figure 2: Visualizations of the Boundary of $\Phi(\mathfrak{I})$. (a) Boundary Created with LPM. (b) Boundary obtained with General Moments. (c) Projection of the Generated Boundary Points based on LPM into the VS-plane. (d) Projection of the Boundary Points generated by General Moments into the VS-plane.

We now compute the product-moment correlation, again in both cases, first between the variance and the skewness of the original 27 stocks, and then between variance and the skewness of the computed boundary points. The results can be found in Table 3.

Table 3
Correlation between Variance and Skewness

	Nr. of Obs.	*Correlation*
LPM	27	-0.95299
	1629	-0.89559
Normal	27	0.17585
	1916	0.01208

Two observations stand out. First, we observe a correlation close to –1 between the lower semi-variance and the lower semi-skewness for both the individual stocks as well as the boundary points. This indicates a strong linear relationship between the lower semi-variance and the lower semi-skewness. Furthermore, the slope of this linear relationship is negative, which explains the left upward orientation of the boundary. In the case of general moments, no significant linear relationship can be detected. We remark here that the same behavior could be noticed in all other experiments that we performed, including the ones on artificial data.

Second, we notice less projected points in the case of lower partial moments. This, however, is merely a consequence of the first observation. Indeed, the more narrow shape produces more infeasibilities when a 2-dimensional grid parallel with the VS-space is projected. We conclude that in the case of lower partial moments, the strong linear relationship between the lower semi-variance and the lower semi-skewness leads to a specific 'stingray-like' shape pointing left upward for $\Phi(\mathfrak{S})$. Both the narrowness and the orientation of $\Phi(\mathfrak{S})$ are responsible for an eventually small strip-like strong efficiency MVS-frontier positioned at the upper left position when viewed in the negative return direction.

5.3. Multidimensional versus Bi-Criteria LMP Models

Figure 3a again shows the visualization when lower partial moments are used and Figure 3b presents the boundary of $\Phi(\mathfrak{S})$ in the case of general moments. Now the gray tone scheme is defined in terms of the skewness dimension (dark gray indicating the highest skewness values). Projection of the generated boundary points based on LPM into the MV-(region ①) and MS-planes (region ②) are shown in Figure 3c, while the same is done for general moments in Figure 3d.

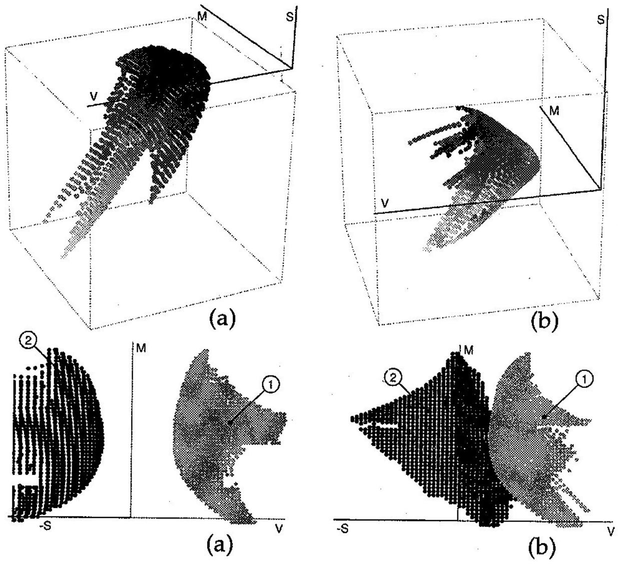

Figure 3: Visualizations of the Boundary of $\Phi(\mathfrak{S})$.
(a) Boundary created with lower partial moments.
(b) Boundary obtained with general moments.
(c) Region : Projection of the boundary points based on LPM into MV-plane. Region : Projection of the Boundary Points based on LPM into MS-plane.
(d) Region: Projection of the Boundary Points Generated by General Moments into MV-plane. Region: Projection of the Boundary Points Generated by General Moments into MS-plane.

Observe that for the LPM, both projections are rather similar in shape (which is less the case for the general moments). Some authors in the literature also exhibit similar pictures for such LPM models (see, e.g., exhibit 4 in Sing and Ong (2000: page 220)). However, given the above demonstration that both mean semi-variance and mean semi-skewness models are special cases of the more general mean semi-variance semi-skewness model, it is important to interpret such projections correctly. While these two-dimensional figures are helpful when investors have only preferences over the dimensions concerned in these bi-criteria models, these are of no use when investors would happen to have preferences over higher order LPM. To choose, for instance, among the portfolios situated on the strip shown in Figure 1, one needs preferences over these three dimensions. Mean-semi-variance and mean-semi-skewness models just reveal the lower and upper boundaries in the skewness dimension of this more general portfolio set and are only compatible with bi-criteria LPM utility functions.

5.4. Analysis of Portfolio Weights

In Figure 4a to Figure 4d, we graphically represent the portfolio weight matrices for the 27 efficient frontier portfolios in both the general and lower partial moments. For each model there is a 27 × 27 squared matrix, whereby the columns are indexed by the stocks for which a shortage function value and a projected portfolio are computed, and the rows are indexed by the number identifying each individual asset. Notice that the ordering of assets on the horizontal axis depends upon their level of risk from low (left) to high (right). Risk is measured by the variance (semi-variance) in general (lower partial) moment models. Detailed rankings per model are available in Appendix. The color code for the weights ranges from deep blue (close to 0) to deep red (close to 1).

One notices in these figures important differences between the models. Contrasting the general moment models (Figure 4a and Figure 4b), one first observes that the resulting frontier portfolios are not ordered similarly in the MV and MVS framework. Second, there is also a difference in the repartition of portfolio weights. In MV, one observes a kind of smooth evolution (gradually increasing weights) corresponding to an increase in risk, while such smooth evolution is not observable in the MVS model.

In the LPM models (Figure 4c and Figure 4d), results vary considerably with respect to what is observed in the general moment MVS framework. However, if one focuses on the differences between the LPM-MV and LPM-MVS models, one can hardly distinguish the added value of the third moment for these 27 assets. Portfolios are ordered in almost the same way in both cases (except for portfolios 12 and 14 that permute in the ranking) and the resulting weights are very similar. Nevertheless, a slight difference can be observed for portfolio 12 (composed with 5 out of 27 assets). This difference is mapped in Figure 4e, whereby portfolios are positioned on the

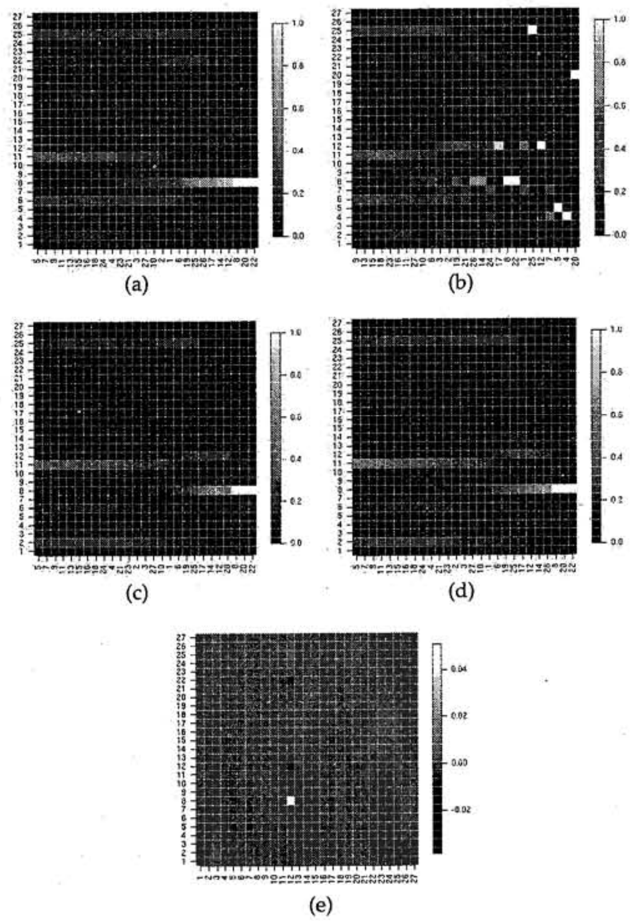

Figure 4: Portfolio Weights: (a) General MV. (b) General MVS. (c) LPM MV. (d) LPM MVS. (e) Differences in weights between LPM MV and LPM MVS.

horizontal axis according to their mere number (irrespective of their risk level). This is done to obtain this figure based upon differences. It is easy to see that five assets are modified substantially in this portfolio 12 when moving between models: weights are increased for asset 12 and 22, while weights are reduced for assets 8 and 14 and 25. None of these assets exhibit high absolute levels of skewness.

To conclude, we focus briefly on the level of diversification for the projected portfolios. We first compute for each model and each optimal portfolio situated onto the frontier, a Herfindhal-Hirschman index. This index is computed using the sum of squared weights for each portfolio: $HH = \sum_i x_i^2$. Notice that this index ranges from 0 to 1 (from extremely diversified to totally specialized portfolios containing only one asset). This index has some known limitations (e.g., favoring highly concentrated portfolios with low diversification level).

The curves representing the Herfindhal-Hirschman index for each of the above models are reported in Figure 5. Apart from the MVS model shown in the upper left window, results for all other models are plotted in the main figure.

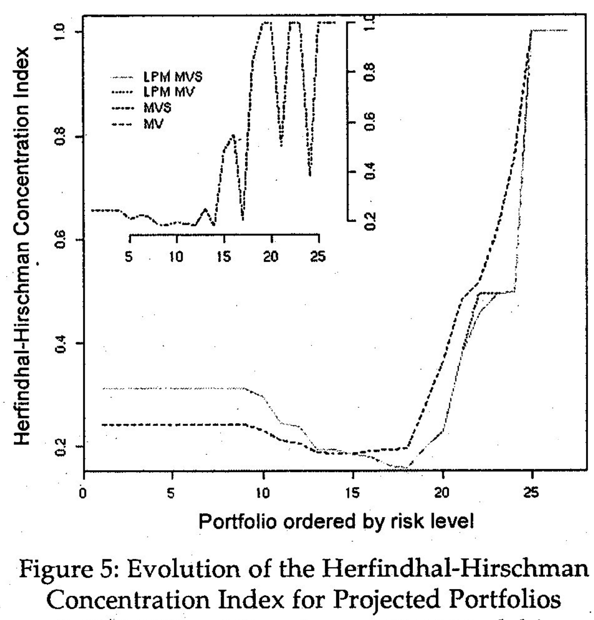

Figure 5: Evolution of the Herfindhal-Hirschman Concentration Index for Projected Portfolios (MV, MVS and Counterpart LPM Models).

Observe that in the main window the curves for both LPM models are very similar and overlap most of the time (except between portfolios 20 and 25). Notice that the 27 portfolios are ordered on the horizontal axis with an increasing level of risk (see comments above). Clearly, for the MV, LPM-MV and LPM-MVS models, increases in risk correspond temporarily with an increase in diversification (until portfolio 15 in MV and portfolio 18 in the LPM-MV and LPM-MVS models). Thereafter, there is a phenomenon of increasing specialization. The relation between risk and diversification/specialization is more non-monotonic and erratic in the MVS model.

6. CONCLUSIONS

The observation that hardly any difference is noticed between the shortage function value obtained in the MV- or the MVS-framework using LPM (in contrast to the general moments case) has triggered several questions.

From our geometric reconstructions, we can draw the preliminary conclusion that in the case of LPM, the strongly efficient LPM MVS-frontier always seems to be a rather small strip-like surface. Its lower bound with respect to the skewness direction is merely the strongly efficient LPM MV-frontier. Its upper bound with respect to the skewness direction is the strongly efficient LPM MS-frontier. This seems to lead to very small differences between the shortage function values computed in the LPM MV-, MS- and MVS-models. Thus, drastic improvements when using the latter more general model should probably not be expected.

Furthermore, the possible small gain when using the LPM MVS-framework must probably be weighted against the shorter computational time needed for the LPM MV- or MS model. This is a practical issue which need not impede further research into the issue.

Finally, although no big differences are observed in the primal framework, this does not imply that the dual utility based assessment is also quite similar. When investors do have preferences over higher LPM (rather than having utility functions compatible with the common bi-criteria LPM models as is traditionally assumed), then small differences in shortage function values need not indicate small utility differences. In the case of LPM MVS, there is also the notion of absolute prudence in addition to the absolute risk aversion. In other words, the decomposition of overall efficiency is important and may lead to different results, since portfolios that are almost identical in terms of portfolio efficiency can yield different utilities.

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APPENDIX

The Table below shows the position of each portfolio in terms of its risk rank (variance for the general moments and semi-variance for the LPM). Rank 1 indicates the portfolio with the lowest risk. One can clearly notice the stability of these rankings for the general MV, LPM MV and LPM MVS models.

Porfolio Risk Rank	*General Moments*		*LPM*		*Porfolio Risk Rank*	*General Moments*		*LPM*	
	MV	*MVS*	*MV*	*MVS*		*MV*	*MVS*	*MV*	*MVS*
1	5	9	5	5	15	10	26	27	27
2	7	13	7	7	16	2	14	10	10
3	9	15	9	9	17	1	24	1	1
4	11	18	11	11	18	6	17	6	6
5	13	23	13	13	19	19	8	19	19
6	15	16	15	15	20	25	22	25	25
7	16	11	16	16	21	26	1	17	17
8	18	27	18	18	22	17	25	14	12
9	24	10	24	24	23	14	12	12	14
10	4	6	4	4	24	12	7	26	26
11	23	3	21	21	25	8	5	8	8
12	21	2	23	23	26	20	4	20	20
13	3	19	2	2	27	22	20	22	22
14	27	21	3	3					

Note :

4 Alternatives to define the lower semi-variance and lower semi-skewness of a portfolio can be considered. For instance, for a given portfolio x, the raw returns can be computed. From the time series obtained, the lower semi-variance and lower semi-skewness can be computed as in (5) and (6). But then, the covariance matrix and coskewness tensor are no longer linked to the lower semi-variance and lower semi-skewness of the portfolio. Consequently, the optimization needed to obtain the shortage function becomes mathematically very complicated. We have opted to avoid these problems.

5 The following mnemonics can be used to retrieve these data with Thomson Financial Datastream: BRL, AIR, CGE, MIDI, BYG, CAN, CGS, CRFR, CSO, CCF, BSN, ORAF, LFG, LVMH, MCL, PGT, PRNT, GOB, SQAF, QTAF, SGE, SDX, LE, CSF, CFP, VAL, EXAF.

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